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Poincaré and Heisenberg quantum dynamical symmetry: Casimir invariant field equations of the quaplectic group

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Abstract. The unitary irreducible representations of a Lie group defines the Hilbert space on which the representations act. If this Lie group is a physical quantum dynamical symmetry group, this Hilbert space is identified with the physical quantum state space. Hermitian representation of the algebra are observables. The eigenvalue equations for the representation of the set of Casimir invariant operators define the field equations of the system. The Poincaré group is the archetypical example with the unitary representations defining the Hilbert space of relativistic particle states and the Klein-Gordon, Dirac, Maxwell equations are obtained from the representations of the Casimir invariant operators eigenvalue equations. The representation of the Heisenberg group does not appear in this derivation. The unitary representations of the Heisenberg group, however, play a fundamental role in nonrelativistic quantum mechanics, defining the Hilbert space and the basic momentum and position commutation relations. Viewing the Heisenberg group as a generalized non-abelian *translation* group, we look for a semidirect product group with it as the normal subgroup that also contains the Poincaré group. The quaplectic group, that is derived from a simple argument using Born's orthogonal metric hypothesis, contains four Poincaré subgroups as well as the normal Heisenberg subgroup. The general set of field equations are derived using the Mackey representation theory for general semidirect product groups. The simplest case of these field equations is the relativistic-oscillator that plays a role in this theory analogous to the Klein-Gordon equation in the Poincaré theory. This theory requires a conjugate relativity principle that bounds forces. Position-time space is no longer an invariant subspace and the quaplectic transformations act on the full nonabelian time-position-momentum-energy space with different observers measuring different position-time subspaces.

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1 Introduction

The unitary irreducible representations of a Lie group are unitary operators on a Hilbert space defining transitions between states and the representation of the algebra are Hermitian operators that are observables. The eigenvalue equations for the representations of the set of Casimir invariant operators define the field equations of the system. The Hilbert space is not given *a priori*, but rather is determined by the unitary representations of the group. We say the Lie group is a dynamical symmetry group if this Hilbert space may be identified with the physical quantum state space with the field equations determining the physical single particle states.

The Poincaré group of special relativity is the archetype dynamical symmetry group. The Poincaré group acts naturally on four dimensional position-time space. Time is not an invariant subspace under the action of this group and is observer dependent. The Poincaré group is a semidirect product of the cover of the Lorentz group and the translation group and its unitary irreducible representations may be obtained from the Mackey theory for semidirect product groups. The field equations defined by the representations define the basic single particle equations of physics; Klein-Gordon, Dirac, Maxwell and so forth. The eigenvalues of the representations of the Casimir operators are spin or helicity and mass.

The Heisenberg group is the semidirect product of two translation groups. It may be regarded as the generalization of a translation group acting on a non-abelian phase space of position and momentum. Again, its representations may be obtained from the Mackey representation theory. The Hilbert space of basic non-relativistic quantum mechanics is defined by the unitary representations of the Heisenberg group. The Lie algebra are the Heisenberg commutators of position and momentum. There is a single Casimir invariant that is the center of the algebra and the field equations of this group are therefore trivial.

It is quite remarkable that the representations of the Poincaré group gives rise to basic equations for single particle states without any reference to the Heisenberg group

that is perceived to be fundamental to quantum mechanics.

The question that this paper addresses is: What is the consequences of defining a group that encompasses both the Poincaré and Heisenberg groups.

The group satisfying this property is a semi-direct product with a generalized *translation* normal subgroup $\mathcal{N}$ that is the Heisenberg group (which itself is the semidirect product of translation groups). The relevant automorphisms of the Heisenberg group are the symplectic group. This together with the requirement for an orthogonal metric leads to the pseudo-unitary group for the homogeneous group $\mathcal{K}$. This group acts on a nonabelian position, time, energy, momentum phase space. It contains four Poincaré subgroups, two associated with special (velocity) relativity and two defining a similar relativity principle that generalizes the concept of force to hyperbolic rotations on the momentum-time and energy-position subspaces. We call this group the *quaplectic* group. It is a subgroup of the inhomogeneous symplectic group, contains four (a quad of) Poincaré groups and has the quantum Heisenberg *translations* as a normal subgroup. The theory embodies Born's reciprocity [1], [2] and the new relativity principle mentioned.

The physical meaning and motivation of the quaplectic group is presented and the unitary irreducible representation of the group and the Hermitian representations of the quaplectic Lie algebra are determined using Mackey's theory of representations of semidirect products. This is the same Mackey theory that is used to determine the unitary irreducible representations of the Poincaré group [3], [4], [5]. As with the Poincaré group, the choice of the group defines the Hilbert space that is identified with quantum particle states for the representations in question. The field equations that are the eigenvalue equations of the representation of the Casimir operators, are obtained and investigated.

2 Dynamical groups in quantum mechanics

The unitary irreducible representations ϱ of a Lie group $\mathcal{G}$ act as unitary operators on a Hilbert space $\mathbf{H}^{\varrho}$. That is, if $g \in \mathcal{G}$

$$\varrho(g) : \mathbf{H}^{\varrho} \to \mathbf{H}^{\varrho} : |\psi\rangle \mapsto \varrho(g)\,|\psi\rangle \qquad (1)$$

where $\varrho(g)^{\dagger} = \varrho(g)^{-1}$. The Hilbert space $\mathbf{H}^{\varrho}$ is determined by the group and the unitary irreducible representation ϱ and so we label it with the representation with the group label implicit. The Lie algebra of the group may be identified with the tangent space of the group $\mathbf{a}(\mathcal{G}) \simeq T_e\mathcal{G}$ and the lift to the representation ϱ' of elements of the algebra $X \in \mathbf{a}(\mathcal{G})$ as Hermitian operators on $\mathbf{H}^{\varrho}$

$$\varrho'(X) : \mathbf{H}^{\varrho} \to \mathbf{H}^{\varrho} : |\psi\rangle \mapsto \varrho'(X)\,|\psi\rangle$$

with $\varrho'(X)^{\dagger} = \varrho'(X)$. The choice of Hermitian representations for the algebra requires a comment. The natural definition is in terms of anti-Hermitian operators that follows from the unitary condition. That is, if $g = e^X$, then

$$\varrho'(g)^{\dagger} = \left(e^{(X)\varrho'}\right)^{\dagger} = e^{\left(\varrho'(X)^{\dagger}\right)} = \varrho(g)^{-1} = e^{-\varrho'(X)}.$$

and so $\varrho'(X)^{\dagger} = -\varrho'(X)$. Defining the Hermitian operator $\tilde{\varrho}'(X) = i\varrho'(X)$, it follows that $\varrho(g) = e^{-i\tilde{\varrho}'(X)}$. Hence forth, we drop the tilde and always uses Hermitian representations of the algebra. Note that an immediate consequence is the appearance of an i in the Lie algebra of the Hermitian representation of the elements $X, Y, Z \in \mathbf{a}(\mathcal{G})$ [6]. That is, if $[X, Y] = Z$ then $[\varrho'(X), \varrho'(Y)] = i\varrho'(Z)$.

Quantum mechanics defines particles states $|\psi\rangle$ to be elements of a Hilbert space $\mathbf{H}$. Observables are Hermitian operators $H = H^{\dagger}$, $H : \mathbf{H} \to \mathbf{H} : |\psi\rangle \mapsto H|\psi\rangle$ and unitary operators $U^{\dagger} = U^{-1}$ defines the evolution of the states. A complete orthonormal basis $|\varphi_x\rangle$ satisfies $\langle\varphi_{\tilde{x}}|\varphi_x\rangle = \delta_{\tilde{x},x}$ and $|\varphi_x\rangle\langle\varphi_{\tilde{x}}| = I$ where I is the identity on $\mathbf{H}$. Observable matrix elements are $h_{\tilde{x},x} = \langle\varphi_{\tilde{x}}|H|\varphi_x\rangle$ with $h_{\tilde{x},x} \in \mathbb{R}$. A unitary operator U transforms orthonormal bases into orthonormal bases $|\tilde{\varphi}_x\rangle = U|\varphi_x\rangle$ and hence preserves probabilities $\langle\tilde{\varphi}_{\tilde{x}}|\tilde{\varphi}_x\rangle = \langle\tilde{\varphi}_{\tilde{x}}|UU^{\dagger}|\varphi_x\rangle = \langle\tilde{\varphi}_{\tilde{x}}|\varphi_x\rangle$. The matrix elements of an observable H are $\langle\tilde{\varphi}_{\tilde{x}}|H|\tilde{\varphi}_x\rangle = \langle\tilde{\varphi}_{\tilde{x}}|U^{\dagger}UHU^{\dagger}U|\varphi_x\rangle = \langle\tilde{\varphi}_{\tilde{x}}|\tilde{H}|\tilde{\varphi}_x\rangle$ with $\tilde{H} = UHU^{\dagger}$.

Casimir invariant operators C_{α} with $\alpha = 1, 2, ...N_c$ are elements of the enveloping algebra of the Lie algebra of the group, $C_{\alpha} \in e(\mathcal{G})$, that commute with all elements of the algebra; $[C_{\alpha}, X] = 0$ for all $X \in \mathbf{a}(\mathcal{G})$. The number of independent Casimir invariant operators is $N_c = N_g - N_r$. N_g is the dimension of the Lie algebra and N_r is its rank. The Hermitian irreducible representation $\varrho'(C_{\alpha})$ have eigenvalues c_{α} that are real constants for a given unitary irreducible representation

$$\varrho'(C_{\alpha})\,|\psi\rangle = c_{\alpha}\,|\psi\rangle \text{ with } |\psi\rangle \in \mathbf{H}^{\varrho}, \quad \alpha = 1, 2...N_c. \qquad (2)$$

These equations in the physical theory are the *field equations* for the dynamical group. The simultaneous solution of these eigenvalue equations define the observable particle states of the theory and the eigenvalues define physically observable, constant properties that are attributed to these particle states.

We will say that $\mathcal{G}$ is a dynamical symmetry of a quantum system if the following conditions are met. First, the Hilbert space $\mathbf{H}^{\varrho}$ determined by the unitary irreducible representations ϱ of a group $\mathcal{G}$ is identical to the Hilbert space $\mathbf{H}$ of the quantum system in question, $\mathbf{H} \simeq \mathbf{H}^{\varrho}$. This means that the (particle) states of the quantum system are states in the unitary irreducible representations of the dynamical group. Unitary operators U defining unitary evolution of the system are given by $U = \varrho(g)$ with corresponding observables $H = \varrho'(X)$. The field equations defining the observable particle state are defined by the eigenvalue equation for the representations of the Casimir invariant operators. It is because these dynamical aspects of the physics arise from the group that we call the group a dynamical symmetry, rather than just a symmetry. This definition of a dynamical group is motivated by the *spectrum generating* or *dynamical* groups as defined by Bohm [7] and reference there-in.

The dynamical groups of interest have the form of a semidirect product Lie group $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{A}$ with $\mathcal{A}$ is the closed normal subgroup and $\mathcal{K}$ the homogeneous group. Furthermore, the Lie groups of interest are real matrix groups that are algebraic. That is, the groups are closed subgroups of $\mathcal{GL}(n,\mathbb{R})$, that are defined by polynomial, or algebraic, constraints. Examples of groups of these type include the special orthogonal, symplectic, unitary, Euclidean, Poincaré, Heisenberg groups as well as, will be shown, the quaplectic group.

The translation group $\mathcal{T}(n+1)$ acts on an abelian position-time manifold $\mathbb{M} \simeq \mathbb{R}^{n+1}$ that may be identified with the physical concept of a flat $n+1$ dimensional position-time space or *space-time*. The usual physical case is $n = 3$.

As noted, the Poincaré group $\mathcal{P}$ is the archetypical dynamical group. It is the cover of the group $\mathcal{E}(1,n) = \mathcal{SO}(1,n)\otimes_s\mathcal{T}(n+1)$ acting on a $n+1$ dimensional position-time space. Then, for the usual physical case $n = 3$, $\mathcal{P} = \overline{\mathcal{E}}(1,3)$. The quantum state space is the Hilbert space of the unitary irreducible representations of the Poincaré group. The Mackey representation theory shows that these Hilbert spaces are

$$\boldsymbol{H}^{\varrho} = \boldsymbol{H}^{\sigma} \otimes \boldsymbol{L}^2(\mathbb{A},\mathbb{C})$$

where $\mathbb{A} \simeq \overline{\mathcal{SO}}(1,n)/\mathcal{K}^{\circ}$ and $\mathcal{K}^{\circ} \simeq \overline{\mathcal{SO}}(n)$, $\overline{\mathcal{E}}(n-1)$, or $\overline{\mathcal{SO}}(1,n-1)$ depending on whether the representations are timelike, null or spacelike. The Hilbert space $\boldsymbol{H}^{\sigma}$ is the corresponding Hilbert space of the unitary irreducible representation of the little group $\mathcal{K}^{\circ}$. For the timelike case, these are finite dimensional and for the physical case $n = 3$, it is just the $2j+1$ dimensional representation spaces of $\mathcal{SU}(2)$ with j half integral. Note that the symmetric spaces $\mathbb{A}$ are the timelike, null and spacelike hyperboloids [5]

There are two Casimir invariant operators in this case with eigenvalue equations for the representations of the Casimir operators acting on the quantum state space of the form (2). The two eigenvalues may be associated with the physical concepts of mass and spin (or helicity) that are constants for each of the irreducible representations. Solution of these eigenvalue equations for the various irreducible representations defines the Klein-Gordon, Dirac, Maxwell and so forth, field equations [4].

3 The quaplectic dynamical group

A dynamical group may also act on the a nonabelian phase space. The Heisenberg group $\mathcal{H}(n) = \mathcal{T}(n)\otimes_s\mathcal{T}(n+1)$ acts as a nonabelian "translation" group on a $2n$ dimensional position-momentum phase space. More generally, $\mathcal{H}(n+1)$ acts on the nonabelian $2n+2$ dimensional position-time-momentum-energy phase space.

In this section we determine the simplest semidirect product group that contains the Poincaré group as a subgroup and also has the Heisenberg group as a normal subgroup. We call the group obtained the *quaplectic* group. We then examine the consequences of it acting as a dynamical symmetry group by determining its unitary irreducible representations and the associated Casimir field equations.

The question investigated in this paper is the group $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{H}(n+1)$ that acts as a dynamical group on the nonabelian phase space analogous to the action of the group $\mathcal{E}(1,n) = \mathcal{SO}(1,n) \otimes_s \mathcal{T}(n+1)$ on the abelian position-time space.

3.1 Group properties

Consider a semidirect product group $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{H}(n+1)$. The first question is to determine the conditions on a subgroup $\mathcal{K}$ that are required in order that this semidirect product can be constructed. As $\mathcal{H}(n+1)$ is a normal subgroup of $\mathcal{G}$, $\mathcal{K}$ must be a subgroup of the automorphisms of $\mathcal{H}(n+1)$. The group product for the elements of the Heisenberg group $h(w,\iota) \in \mathcal{H}(n+1)$ may be written

$$\begin{aligned} &h(\tilde{w},\tilde{\iota}) \cdot h(w,\iota) = h(w+\tilde{w}, \iota+\tilde{\iota}+\tilde{w}\cdot\zeta\cdot w) \\ &h^{-1}(w,\iota) = g(-w,-\iota) \end{aligned} \tag{3}$$

where $g(0,0)$ is the identity element and $w \in \mathbb{R}^{2(n+1)}$, $\iota \in \mathbb{R}$ and $\tilde{w}\cdot\zeta\cdot w = \tilde{w}^a\zeta_{a,b}w^b$ with the indices $a,b,.. = 0,1,...n$ and $i,j,.. = 1,...n$. This index convention is always assumed to be the case unless explicitly noted otherwise. The components of the symplectic metric ζ in these canonical coordinates is the $2(n+1)\times 2(n+1)$ matrix

$$\zeta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is $(n+1)\times(n+1)$ identity matrix. Therefore, while the Heisenberg group has no implicit concept of an orthogonal metric, it does have implicit in its definition a symplectic structure. The Heisenberg group is a semidirect product $\mathcal{H}(n+1) \simeq \mathcal{T}(n+1)\otimes_s\mathcal{T}(n+2)$ and it is a matrix group that is algebraic

$$h(w,\iota) \simeq \begin{pmatrix} I & 0 & w \\ w\cdot\zeta & 1 & \iota \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra is spanned by the basis $\{W_\mu\}$ with $\mu,\nu = 0,...n,\tilde{0},...\tilde{n}$ that satisfies the Lie algebra relations

$$[W_\mu, W_\nu] = \zeta_{\mu,\nu} I.$$

If we identify $\{W_\mu\} = \{T, P_i, -E, Q_i\}$ then $[P_i, Q_j] = \delta_{i,j} I$ and $[T,E] = -I$. (We are using natural units with $\hbar = 1$. Units are discussed further shortly.)

The action $\varsigma_a h \doteq a\cdot h\cdot a^{-1}$ of the Heisenberg group on itself are the automorphisms

$$\varsigma_{h(\tilde{w},\tilde{\iota})} h(w,\iota) = h(w, \iota + 2w\cdot\zeta\cdot\tilde{w}), \tag{4}$$

Elements of the complete group of linear automorphisms [8] of $\mathcal{H}(n+1)$ have the action

$$\varsigma_{a_\pm(\tilde{\varepsilon},\tilde{A},\tilde{w},\tilde{\iota})} h(w,\iota) = h(\tilde{\varepsilon}\tilde{A}w, \pm\varepsilon^2(\iota + 2w\cdot\zeta\cdot\tilde{w})), \tag{5}$$

where $a(1,A,0,0)\in\mathcal{S}p(2n+2)$, $a(\epsilon,I,0,0)\in\mathcal{A}b(1)$ is the one parameter, ϵ, abelian group and the discrete symmetry is $a_\pm\in\mathcal{D}_2$. Thus, the group of linear automorphisms [8] $\mathcal{A}ut(n+1)$ of $\mathcal{H}(n+1)$has the form

$$\begin{aligned}\mathcal{A}ut(n+1)&=\mathcal{D}_2\otimes(\mathcal{A}b(1)\otimes\mathcal{S}p(2n+2))\otimes_s\mathcal{H}(n+1)\\&=\mathcal{D}_2\otimes\mathcal{A}b(1)\otimes_s\mathcal{HS}p(n+1)\end{aligned}$$

where $\mathcal{HS}p(n)\doteq\mathcal{S}p(2n)\otimes_s\mathcal{H}(n)$. A general element of the continuous automorphisms $\mathcal{A}ut(n+1)$ may be written

$$a(\varepsilon,A,w,\iota)=a(\varepsilon,I,0,0)\cdot a(1,A,0,0)\cdot a(1,I,w,\iota)$$

and a general element $a(\varepsilon,A,w,\iota)$ is represented by the matrix group

$$a(\varepsilon,A,w,\iota)\simeq\begin{pmatrix}A&0&A\cdot w\\ \varepsilon w\cdot\zeta&\varepsilon&\varepsilon\iota\\0&0&\varepsilon^{-1}\end{pmatrix},$$

where $\varepsilon\in\mathbb{R}\backslash\{0\}$ and $A\in\mathcal{S}p(2n+2)$defined by $A_1..A_4$ real $(n+1)\times(n+1)$ matrices satisfying

$$A=\begin{pmatrix}A_1&A_3\\A_4&A_2\end{pmatrix},\quad A^{-1}=\begin{pmatrix}{}^tA_2&-{}^tA_3\\-{}^tA_4&{}^tA_1\end{pmatrix}.$$

The group composition law and inverse may be computed directly from this faithful matrix representation, or abstractly by using the property Heisenberg group is a normal subgroup and therefore

$$\begin{aligned}&a(\tilde{\varepsilon},\tilde{A},\tilde{w},\tilde{\iota})\cdot a(\varepsilon,A,w,\iota)\\&=a(\tilde{\varepsilon},\tilde{A})\cdot a(\varepsilon,A)\cdot a(\varepsilon,A)^{-1}\cdot h(\tilde{w},\tilde{\iota})\cdot a(\varepsilon,A)\cdot h(w,\iota)\\&=a(\tilde{\varepsilon}\varepsilon,\tilde{A}\cdot A,w+\varepsilon^{-1}A^{-1}\cdot\tilde{w},\iota+\varepsilon^{-2}\tilde{\iota}+\varepsilon^{-1}\tilde{w}\cdot\zeta\cdot A\cdot w)\end{aligned}\tag{6}$$

The semidirect product group $\mathcal{G}$ with $\mathcal{H}(n+1)$ as a normal subgroup must, in general, be a subgroup of $\mathcal{A}ut(n+1)$. In this initial investigation, we do not consider the effects of $\mathcal{A}b(1)$ nor the discrete automorphisms further and so consider $\mathcal{G}$ to be a subgroup of $\mathcal{HS}p(n+1)$. Elements g of the $\mathcal{HS}p(n+1)$ subgroup are $g(A,w,\iota)\simeq a(1,A,w,\iota)$.

Consider the particular automorphism $a(1,\Upsilon^\circ,0,0)\in\mathcal{A}ut(n+1)$, and therefore $\Upsilon^\circ\in\mathcal{S}p(2n+2)$ that is the particular element that has the property $\hat{w}=\Upsilon^\circ\cdot w$ with $\hat{w}=(-w^{\tilde{0}},w^1,..w^n,w^0,w^{\tilde{1}}...w^{\tilde{n}})$. Applying the automorphism to the full $\mathcal{HS}p(n+1)$ group puts the components of the symplectic metric into the following form with the associated symplectic inverse condition

$$\hat{\zeta}=\begin{pmatrix}0&\eta\\-\eta&0\end{pmatrix},\quad\hat{A}^{-1}=\begin{pmatrix}\eta\cdot{}^t\hat{A}_2\cdot\eta&-\eta\cdot{}^t\hat{A}_3\cdot\eta\\-\eta\cdot{}^t\hat{A}_4\cdot\eta&\eta\cdot{}^t\hat{A}_1\cdot\eta\end{pmatrix}.\tag{7}$$

where η is $(n+1)\times(n+1)$ diagonal matrix with diagonal $(-1,1,..1)$ and $\eta^{-1}=\eta$. Note that the $\hat{A}_\alpha$ are an intertwined combination of the components of the A_α.

The Lie algebra is

$$\left[\hat{W}_\mu,\hat{W}_\nu\right]=\hat{\zeta}_{\mu,\nu}I,\quad\mu,\nu=0,...n,\tilde{0},...\tilde{n}$$

where now we identify $\{\hat{W}_\mu\}=\{E,P_i,T,Q_i\}$. Then, again, $[P_i,Q_j]=\delta_{i,j}I$ and $[T,E]=-I$. That the algebra is in, fact identical to that of the $\{W_\mu\}$, emphasizes again that this group is isomorphic to the original group as required under the action of the automorphism $a(1,\Upsilon^\circ,0,0)$. Again, it is emphasized that $\mathcal{HS}p(n+1)$ has no concept of an orthogonal metric and that "$\mathcal{HS}p(1,n)\simeq\mathcal{HS}p(n+1)$". We use this form of the $\mathcal{HS}p(n+1)$ group and drop the tilde in what follows.

While $\mathcal{HS}p(n+1)$ has no concept of an orthogonal metric, the Poincaré group acting on the position-time and the Poincaré group acting on energy-momentum spaces have Casimir invariant operators that define pseudo-orthogonal metric structures

$$-T^2+\frac{1}{c^2}Q^2,\qquad\qquad P^2-\frac{1}{c^2}E^2.$$

Born [1],[2] based on a reciprocity principle, argued that these metrics combine into a single metric on $2(n+1)$ dimensional phase space

$$-T^2+\frac{1}{c^2}Q^2+\frac{1}{b^2}\left(P^2-\frac{1}{c^2}E^2\right).\tag{8}$$

b is a new universal dimensional physical constant with units of force that we shall discuss shortly. The group with both an orthogonal and symplectic structure is the unitary group $\mathcal{U}(1,n)\simeq\mathcal{O}(2,2n)\cap\mathcal{S}p(2n+2)$. This group leaves invariant the symplectic structure as well as the orthogonal metric.

As $\mathcal{U}(1,n)$ is a subgroup of $\mathcal{S}p(2n+2)$ we may define a semidirect product group $\mathcal{Q}(1,n)$ that has a dimension $N=(n+2)^2$, that we name the quaplectic group, by

$$\begin{aligned}\mathcal{Q}(1,n)&=\mathcal{U}(1,n)\otimes_s\mathcal{H}(n+1)\\&=\mathcal{SU}(1,n)\otimes_s\mathcal{O}s(n+1)\end{aligned}\tag{9}$$

where $\mathcal{O}s(n)=\mathcal{U}(1)\otimes_s\mathcal{H}(n)$. The orthogonal condition of $\mathcal{O}(2,2n)$ implies that, in addition to the symplectic inverse condition, (3.7), the A must also satisfy the condition

$$A^{-1}=\begin{pmatrix}\eta\cdot{}^tA_1\cdot\eta&\eta\cdot{}^tA_4\cdot\eta\\\eta\cdot{}^tA_3\cdot\eta&\eta\cdot{}^tA_2\cdot\eta\end{pmatrix}$$

from which it follows that $A_1=A_2=\Lambda$ and $A_3=-A_4=M$ and imposes the conditions that $\Lambda^{-1}=\eta\cdot{}^t\Lambda\cdot\eta$ and $M^{-1}=-\eta\cdot{}^tM\cdot\eta$ and so the element of a faithful matrix representation of $\mathcal{H}(n+1)$ has the form

$$g(\Lambda,M,x,y,\iota)\simeq\begin{pmatrix}\Lambda&M&0&\Lambda\cdot x+M\cdot y\\-M&\Lambda&0&-M\cdot x+\Lambda\cdot y\\-y&x&1&\iota\\0&0&0&1\end{pmatrix}$$

where $w=(x,y)$ and $x,y\in\mathbb{R}^{n+1}$. The group multiplication law for $\mathcal{Q}(1,n)$ may be written in a complex notation by defining $z=x+iy$, $z\in\mathbb{C}^{n+1}$ and $\Upsilon=\Lambda+iM$. Then

$$\begin{aligned}&g(\tilde{\Upsilon},\tilde{z},\tilde{\iota})\cdot g(\Upsilon,z,\iota)\\&=g(\tilde{\Upsilon}\cdot\Upsilon,\tilde{z}+\Upsilon\cdot z,\iota+\tilde{\iota}+\tfrac{i}{2}\left(\overline{z}\cdot\eta\cdot\tilde{z}-\overline{\tilde{z}}\cdot\eta\cdot z\right)),\\&g^{-1}(\Upsilon,z,\iota)=g(\Upsilon^{-1},-\Upsilon^{-1}\cdot z,-\iota).\end{aligned}\tag{10}$$

An element $g \in \mathcal{Q}(1,n)$ may be realized by the $(n+1)\times(n+1)$ complex matrices

$$g(\Upsilon, z, \iota) \simeq \begin{pmatrix} \Upsilon & 0 & \Upsilon\cdot z \\ \overline{z} & 1 & \iota \\ 0 & 0 & 1 \end{pmatrix}.$$

The inner automorphisms of the group are

$$\varsigma_{g(\tilde{\Upsilon},\tilde{z},\tilde{\iota})} g(\Upsilon, z, \iota) \simeq g(\tilde{\Upsilon}\cdot\Upsilon\cdot\tilde{\Upsilon}^{-1}, \tilde{z}+\tilde{\Upsilon}\cdot z, \iota+\iota+i(\overline{\tilde{z}}\cdot\eta\cdot\tilde{z}-\overline{z}\cdot\eta\cdot\tilde{z})).$$

If $M=0$, we obtain the $\mathcal{SO}(1,n)\otimes_s\mathcal{H}(n+1)$ subgroup with elements of the form $g(\Lambda, z, \iota)$.

3.2 Lie algebra and Casimir invariant operators

A general element of the Canonical algebra is $Z+A$ with Z and element of the algebra of $\mathcal{U}(1,n)$ and A an element of the Heisenberg algebra of $\mathcal{H}(n+1)$,

$$\begin{array}{l} Z = \phi^{a,b}M_{a,b} + \varphi^{a,b}L_{a,b}, \\ A = \iota I + x^a X_a + y^a Y_a. \end{array}$$

with the parameters all real. These generators satisfy

$$\begin{array}{l}
[L_{a,b}, L_{c,d}] = -L_{b,d}\eta_{a,c} + L_{b,c}\eta_{a,d} + L_{a,d}\eta_{b,c} - L_{a,c}\eta_{b,d} \\
[L_{a,b}, M_{c,d}] = -M_{b,d}\eta_{a,c} - M_{b,c}\eta_{a,d} + M_{a,d}\eta_{b,c} + M_{a,c}\eta_{b,d} \\
[M_{a,b}, M_{c,d}] = -L_{b,d}\eta_{a,c} - L_{b,c}\eta_{a,d} - L_{a,d}\eta_{b,c} - L_{a,c}\eta_{b,d} \\
[L_{a,b}, X_c] = -X_b\eta_{a,c} + X_a\eta_{b,c} \\
[L_{a,b}, Y_c] = -Y_b\eta_{a,c} + Y_a\eta_{b,c} \\
[M_{a,b}, X_c] = -Y_b\eta_{a,c} - Y_a\eta_{b,c} \\
[M_{a,b}, Y_c] = X_b\eta_{a,c} + X_a\eta_{b,c} \\
[X_a, Y_b] = I\eta_{a,b}
\end{array} \tag{11}$$

Clearly both the generators $\{L_{a,b}, X_c\}$ and $\{L_{a,b}, Y_c\}$ define the algebras of Poincaré subgroups. The second order Casimir invariant operator is

$$C_2 = \frac{1}{2}\eta^{a,b}(X_aX_b + Y_aY_b) + \frac{1}{2}(n+1)I - IU.$$

As $I = C_1$ is a Casimir operator, and any linear combinations of Casimir operators is also one, this term may be dropped. The algebra of $\mathcal{Q}(1,n) = \mathcal{U}(1,n)\otimes\mathcal{H}(n+1)$ may also be written in a complex form by defining

$$A_a^{\pm} = \frac{1}{\sqrt{2}}(X_a \mp iY_a), \quad Z_{a,b} = \frac{1}{2}(M_{a,b} - iL_{a,b}).$$

It follows directly that the Lie algebra relations take the more condensed form

$$\begin{array}{l} [Z_{a,b}, Z_{c,d}] = i(Z_{c,b}\eta_{a,d} - Z_{a,d}\eta_{b,c}), \\ \left[A_a^+, A_b^-\right] = i\eta_{a,b}I,\ [Z_{a,b}, A_c^{\pm}] = \mp i\eta_{a,c}A_b^{\pm} \end{array} \tag{12}$$

where as usual, the indices $a,b = 0,1,...n$. $\{Z_{a,b}\}$ span the algebra of $\mathcal{U}(1,n)$ and $\{A_c^{\pm}, I\}$ span the algebra of $\mathcal{H}(1,n)$.

The compact case of $\mathcal{Q}(n) = \mathcal{U}(n)\otimes\mathcal{H}(n)$ is immediately obtained by restricting the indices a,b to $i,j = 1,..n$. The generators $\{Z_{i,j}, A_k^{\pm}, I\}$ span the algebra of the subgroup $\mathcal{Q}(n)$ of $\mathcal{Q}(1,n)$, where we note that $\eta_{i,j} = \delta_{i,j}$.

Define $U \doteq \eta^{a,b}Z_{a,b}$. This is the $\mathcal{U}(1)$ generator in the decomposition $\mathcal{Q}(1,n) = \mathcal{SU}(1,n)\otimes_s\mathcal{O}s(1,n)$ with $\mathcal{O}s(1,n) = \mathcal{U}(1)\otimes_s\mathcal{H}(1,n)$. The corresponding generators of the algebra of $\mathcal{SU}(1,n)$ are $\{\hat{Z}_{a,b}\}$

$$Z_{a,b} = \hat{Z}_{a,b} + \frac{1}{n+1}U\eta_{a,b} \tag{13}$$

Note that $\eta^{a,b}\hat{Z}_{a,b} = 0$ and that the $Z_{a,b}$ and $\hat{Z}_{a,b}$ commute with U and the $\hat{Z}_{a,b}$ satisfy the same commutation relations as defined above in (12).

For $\mathcal{Q}(1,n)$, $N_c = n+2$ and the $n+2$ Casimir invariant operators are [9]

$$\begin{array}{l} C_1 = I, \qquad C_2 = \eta^{a_1,a_2}W_{a_1,a_2}, \qquad ... \\ C_{2\beta} = \eta^{a_1,a_{2\beta}}...\eta^{a_{2\beta-2},a_{2\beta-1}}W_{a_1,a_2}...W_{a_{2\beta-1},a_{2\beta}}, \end{array} \tag{14}$$

with $\beta = 1,.n+1$ and $W_{a,b} \doteq A_a^+A_b^- - IZ_{a,b}$. Note that the second order invariant is of the form

$$C_2 = \eta^{a,b}A_a^+A_b^- - IU = A^2 - IU, \tag{15}$$

where U is the generator of the algebra of $\mathcal{U}(1)$ defined above. The commutation relations for the $W_{a,b}$ are

$$\begin{array}{l} [Z_{a,b}, W_{c,d}] = i(\eta_{a,d}W_{b,c} - \eta_{b,c}W_{d,a}) \\ [A_c^{\pm}, W_{a,b}] = 0 \end{array} \tag{16}$$

and therefore $W_{c,d}$ are Heisenberg *translation* invariant. It is important to note that both of the terms in $W_{a,b}$ are required in order for the commutator to vanish with $A_c^{\pm}$. The $W_{c,d}$ commutators with $Z_{a,b}$ are the same as $Z_{c,d}$. The Casimir invariants of $\mathcal{U}(1,n)$ are [10]

$$D_\beta = \eta^{a_1,a_{2\beta}}...\eta^{a_{2\beta-2},a_{2\beta-1}}Z_{a_1,a_2}...Z_{a_{2\beta-1},a_{2\beta}}, \tag{17}$$

where $\beta = 1,...,n+1$. Therefore, (3.14) are invariant unitary $\mathcal{U}(1,n)$ rotations and, as the $W_{a,b}$ have already been established to be Heisenberg translational invariant, it follows that they are Casimir invariants of $\mathcal{Q}(1,n)$. Note also that it follows immediately that

$$[D_\beta, D_\alpha] = 0,\ [D_\beta, I] = 0, [D_\beta, C_{2\alpha}] = 0, \tag{18}$$

with $\alpha,\beta = 1,...n+1$. As an aside that will be useful, we note also that $\mathcal{O}s(n) = \mathcal{U}(1)\otimes_s\mathcal{H}(n)$ is rank 2 for all n. The two Casimir invariants $\tilde{C}_1$ and $\tilde{C}_2$ are

$$\tilde{C}_1 = C_1 = I, \qquad \tilde{C}_2 = C_2 = A^2 - IU. \tag{19}$$

3.3 Physical interpretation of the quaplectic group

The Poincaré group is a dynamical symmetry of Minkowski space $\mathbb{M}^{1,n} = \mathcal{E}(1,n)/\mathcal{SO}(1,n)$. The generators $\{Y_a\} = \{E, P_i\}$ of the translation group on position-time space are identified with energy and momentum. In the quantum theory, the unitary representations of the Poincaré group

are required and the corresponding Hermitian representation of these generators define the observable energy and momentum degrees of freedom.

The quaplectic group acts on the nonabelian phase space $\mathbb{Q}^{1,n} = \mathcal{Q}(1,n)/\mathcal{SU}(1,n)$. Again the generators $\{Y_a\}$ are the energy and momentum but the Heisenberg algebra also includes the $\{X_a\} = \{T, Q_i\}$ that are the time and position degrees of freedom. Again, in the quantum theory, the Hermitian representations of these generators defines these observable degrees of freedom.

Macroscopic physics assigns different dimensions to the time, energy, position and momentum degrees of freedom that are usually defined in terms of the constants $c, \hbar$ and G. Instead of G, we choose to use the constant b introduced in Section (8) that has units of force and G may be defined in terms of it as $G = \alpha_G c^4/b$ where α_G is just the dimensionless gravitational coupling constant to be determined by theory or experiment. Planck scales λ_α of time, position, momentum and energy may be defined in terms of these three constants as

$$\lambda_t = \sqrt{\hbar/bc}, \lambda_q = \sqrt{\hbar c/b}, \lambda_p = \sqrt{\hbar b/c}, \quad \lambda_e = \sqrt{\hbar bc}$$

If $\alpha_G = 1$, these are just the usual Planck scales. The Lorentz subgroup of the Poincaré group transforms position and time degrees of freedom into each other. This mixing is *real* in the sense that observers in a frame measure *position* and measure *time* of the position-time space differently with the usual time dilation and length contraction effects when measurements are compared between frames. The constant c is required to convert units of time into position and vice versa. This is no different that if we measured the x direction in meters and the y direction in feet so that every time a rotation was applied, a constant converting meters into feet and vice versa is required. Simply by choosing the right units, this constant is eliminated and so it is also with c, in *natural* units $c = 1$.

The quaplectic group transforms all of the degrees of freedom, position, time, energy and momentum into each other. A special case of this is the usual Lorentz transformations of special (velocity) relativity acting on the momentum-energy and position-time subspaces. The full set of transformations cause a mixing of all the degrees of freedom that are as *real* as the Lorentz velocity subset. This mixing is *real* in the sense that observers in a frame measure *position, time, energy* and *momentum* of the nonabelian position-time-energy-momentum space differently with generalizations of the usual time dilation and length contraction effects when measurements are compared between frames. Position-time and momentum-energy are not invariant subspaces under the action of the group. The constants b, c and $\hbar$ are required to describe the conversion of units required in the mixing rather than just c as in the Poincaré case. Again, by choosing natural units with $c = b = \hbar = 1$, these constants are eliminated. We generally use natural units unless specifically noted.

This is made considerably more subtle than the Poincaré case by the fact that the underlying manifold is now nonabelian. The measurements, in fact, require the quantum theory involving the Hermitian representation of the algebra. A frame is now a particle state which is identified with a state in a unitary irreducible representation of the quaplectic group. This representation theory is discussed in the section that follows.

The transformation of the Heisenberg generators between frames is mathematically the group automorphisms of the algebra. That is, for $A \in \boldsymbol{a}(\mathcal{H}(1,n))$ and $\Upsilon \in \mathcal{U}(1,n)$ with $\Upsilon = e^Z$, the transformed generators $\tilde{A}$ are

$$\varsigma'_\Upsilon : A \mapsto \tilde{A} = \Upsilon \cdot A \cdot \Upsilon^{-1} = e^Z \cdot A \cdot e^{-Z} = A + [Z, A] + \quad (20)$$

With $n = 3$, define $J_i = \epsilon_k^{j,k} L_{j,k}$, $K_i = L_{0,i}$, $N_i = M_{0,i}$ and $R = M_{0,0}$ where $\{L_{a,b}, M_{a,b}\}$ are the generators of $\mathcal{U}(1,n)$ defined in (11). A general element of the quaplectic algebra is $Z + A$ where

$$\begin{aligned} Z &= \beta^i K_i + \gamma^i N_i + \alpha^i J_i + \theta^{i,j} M_{i,j} + \vartheta R, \\ A &= \tfrac{t}{\lambda_t} T + \tfrac{e}{\lambda_e} E + \tfrac{q^i}{\lambda_q} Q_i + \tfrac{p^i}{\lambda_p} P_i + \iota I. \end{aligned} \quad (21)$$

In this discussion, the Heisenberg algebra degrees of freedom take on dimensional scales corresponding to the different units of measurement for time, energy, position, and momentum. The infinitesimal transformations $\tilde{A} = A + [Z, A]$ for the basis are

$$\begin{aligned} \tilde{T} &= T + \beta^i Q_i/c + \gamma^i P_i/b + \vartheta E/cb, \\ \tilde{E} &= E - c\gamma^i Q_i + b\beta^i P_i - bc\vartheta T, \\ \tilde{Q}_i &= Q_i + \epsilon_{i,j}^k \alpha^j Q_k + c\beta^i T - \gamma^i E/b + c\theta^{i,j} P_j/b, \\ \tilde{P}_i &= P_i + \epsilon_{i,j}^k \alpha^j Q_k + \beta^i E/c + b\gamma^i T - b\theta^{i,j} Q_j/c. \end{aligned} \quad (22)$$

And, with $\alpha^i = \theta^{i,j} = \vartheta = 0$, the *pure boost* finite transformations are

$$\begin{aligned} \tilde{T} &= \cosh\omega T + \tfrac{\sinh\omega}{\omega}\left(\tfrac{\beta^i}{c} Q_i + \tfrac{\gamma^i}{b} P_i\right), \\ \tilde{E} &= \cosh\omega E + \tfrac{\sinh\omega}{\omega}\left(-b\gamma^i Q_i + c\beta^i P_i\right), \\ \tilde{Q}_i &= Q_i + \tfrac{\cosh\omega - 1}{\omega^2}\omega^{i,j} Q_j + \tfrac{\sinh\omega}{\omega}\left(c\beta^i T - \tfrac{\gamma^i}{b} E_i\right), \\ \tilde{P}_i &= P_i + \tfrac{\cosh\omega - 1}{\omega^2}\omega^{i,j} P_j + \tfrac{\sinh\omega}{\omega}\left(b\gamma^i T + \tfrac{\beta^i}{c} E\right) \end{aligned} \quad (23)$$

where $\omega^{i,j} = \beta^i\beta^j + \gamma^i\gamma^j$ and $\omega = \sqrt{\delta_{i,j}\omega^{i,j}}$. Note immediately that if $\gamma^i = 0$ these are the equations for the usual pure Lorentz velocity boost from basic special relativity for inertial (non-interacting) frames

$$\begin{aligned} \tilde{T} &= \cosh\beta T + \tfrac{\sinh\beta}{\beta}\tfrac{\beta^i}{c} Q_i, \\ \tilde{Q}_i &= Q_i + \tfrac{\cosh\beta - 1}{\beta^2}\beta^i\beta^j Q_j + \tfrac{\sinh\beta}{\beta} c\beta^i T, \\ \tilde{E} &= \cosh\beta E + \tfrac{\sinh\beta}{\beta} c\beta^i P_i, \\ \tilde{P}_i &= P_i + \tfrac{\cosh\beta - 1}{\beta^2}\beta^i\beta^j P_j + \tfrac{\sinh\beta}{\beta}\tfrac{\beta^i}{c} E. \end{aligned} \quad (24)$$

Conversely if $\beta^i = 0$, these equations reduce to

$$\begin{aligned} \tilde{T} &= \cosh\gamma T + \tfrac{\sinh\gamma}{\gamma}\tfrac{\gamma^i}{b} P_i, \\ \tilde{P}_i &= P_i + \tfrac{\cosh\gamma - 1}{\gamma^2}\gamma^i\gamma^j P_j + \tfrac{\sinh\gamma}{\gamma} b\gamma^i T, \\ \tilde{E} &= \cosh\gamma E + \tfrac{\sinh\gamma}{\gamma} b\gamma^i Q_i, \\ \tilde{Q}_i &= Q_i + \tfrac{\cosh\gamma - 1}{\gamma^2}\gamma^i\gamma^j Q_j - \tfrac{\sinh\gamma}{\gamma}\tfrac{\gamma^i}{b} E. \end{aligned} \quad (25)$$

and consequently for this special case, we have a conjugate set of pure Lorentz boost transformations describing boosts between frames with relative rate of change

of momentum, that is force, but with negligible velocity. Clearly this an asymptotic case but it must also be emphasized that a purely, non interacting particle, is also an asymptotic case, particularly in the Planck regime where these effects will be manifest. This limit illustrates the bounding of force that is related to proposed acceleration bounding theories [11],[12],[13] or minimum length [14]. These asymptotic cases are just intended to highlight the properties of the general equations (22).

The quaplectic group has two different $\mathcal{SO}(1,n)\otimes_s \mathcal{H}(1,n)$ subgroups generated by the sets of generators

$$\{J_i, K_i, E, P_i, T, Q_i, I\}\,, \qquad \{J_i, N_i, E, P_i, T, Q_i, I\}$$

These are the infinitessimal generators that, using by (20), may be exponentiated to the finite transformation equations (24) and (25) respectively. (Note in the finite equations given, the rotation parameters α^i associated with the generators J_i are zero. Each of these in turn has two distinct Poincaré subgroups for the total of four distinct Poincaré subgroups of the quaplectic group generated by the sets of generators

$$\begin{array}{l}\{J_i, K_i, E, P_i\}\,,\ \{J_i, K_i, T, Q_i\}\\ \{J_i, N_i, E, Q_i\}\,,\ \{J_i, N_i, T, P_i\}\end{array}$$

The Hermitian representation of the translation generators of only one of the translation subgroups of these four Poincaré groups can be diagonalized at a time. This may be denoted by a simple quad, where the representations of the generators on only one face may be simultaneously diagonalized.

$$\begin{array}{ccc}\varrho'(T) & \leftrightarrow & \varrho'(Q_i)\\ \updownarrow & & \updownarrow\\ \varrho'(P_i) & \leftrightarrow & \varrho'(E)\end{array}$$

Under the special case of pure velocity boosts, this nonabelian space breaks into invariant abelian time-position and momentum-position subspaces on which the transformations (24) act. Likewise, in the special case of pure force boosts, this nonabelian space breaks into invariant abelian time-momentum and position-energy subspaces on which the transformations (25) act.

It is important to emphasized that in general, all degrees of freedom of this non abelian space mix as one transforms from one frame to another. The physical theory quantum theory that arises from the unitary representations of the quaplectic group has states for interacting particles, not just the asymptotic free particle states. The unitary representations of the quaplectic group transform one state into one another and, in so doing, all of the degrees of freedom, time, position, energy and momentum mix.

This leads to a a note about the name *quaplectic* of the group. The literal meaning of the English word *qua* is similar to the preposition *as*, ' in the role or character of' and *-plectic* has origins in *pleat* 'to fold on itself'. So the literal meaning of *quaplectic* is 'in the character of folding on itself'. *Quaplectic* is also derived from the origins of the group in the *sym-plectic* group with the *quantum* Heisenberg 'translation' subgroup and *qua-d* (as in *quad* Poincaré subgroups and the nonabelian *quad*) connotations.

4 Unitary representations of the quaplectic group

This section reviews the Mackey theory for unitary irreducible representations of the semidirect product group and applies it to the quaplectic group and algebra. The Heisenberg group is a normal subgroup of the quaplectic group. As the representations of the normal group are required by the Mackey theory, the representations of the Heisenberg group are briefly reviewed. As it is, in turn a semidirect product group, the Mackey theory is again applicable.

4.1 Unitary irreducible representations of semidirect product groups

The problem of determining the unitary irreducible representations of a general class of semidirect product groups has been solved by Mackey [3]. A sufficient condition for the Mackey representation theory to apply is that the groups are matrix groups that are algebraic. The Mackey theorems are reviewed in [5] and briefly summarized here. In addition, the manner in which the results lift to the algebra is given as they are required for the determination of the field equations.

Suppose that $\mathcal{A}$ and $\mathcal{N}$ are matrix groups that are algebraic with unitary irreducible representations ξ and σ on the respective Hilbert spaces $\mathbf{H}^\xi$ and $\mathbf{H}^\sigma$. Then for $a\in\mathcal{A}$, and $k\in\mathcal{K}$

$$\begin{array}{l}\xi(a):\mathbf{H}^\xi\to\mathbf{H}^\xi:|\phi\rangle\mapsto\left|\tilde{\phi}\right\rangle=\xi(a)\,|\phi\rangle\,,\\ \sigma(k):\mathbf{H}^\sigma\to\mathbf{H}^\sigma:|\varphi\rangle\mapsto|\tilde{\varphi}\rangle=\sigma(k)\,|\varphi\rangle\,.\end{array}$$

The general problem is to determine the unitary irreducible representations ϱ, and the Hilbert space $\mathbf{H}^\varrho$ on which it acts, of the semidirect product $\mathcal{G}=\mathcal{K}\otimes_s\mathcal{A}$, $\varrho(g):\mathbf{H}^\varrho\to\mathbf{H}^\varrho:|\psi\rangle\mapsto|\tilde{\psi}\rangle=\varrho(g)|\psi\rangle$.

The Mackey theorems state that these unitary irreducible representations ϱ may be constructed by first determining the representations ϱ° of the stabilizer groups, $\mathcal{G}^\circ\subseteq\mathcal{G}$ and then using an induction theorem to obtain the representations on the full group $\mathcal{G}$. A sufficient condition for the Mackey group to apply is that $\mathcal{G}$,$\mathcal{K}$ and $\mathcal{A}$ are matrix groups that are algebraic in the sense that they are defined by polynomial constraints on the general linear groups.

The stabilizer group $\mathcal{G}^\circ=\mathcal{K}^\circ\otimes_s\mathcal{A}$ where $\mathcal{K}^\circ$ is defined for each of the *orbits.* These orbits are defined by the natural action of elements $k\in\mathcal{K}$ on the unitary dual $\hat{\mathcal{A}}$ of $\mathcal{A}$. The action defining the orbits is $k:\hat{\mathcal{A}}\to\hat{\mathcal{A}}:\xi\mapsto\tilde{\xi}=k\xi$ where $(k\xi)(a)=\xi(k\cdot a\cdot k^{-1})$ for all $a\in\mathcal{A}$. The little groups $\mathcal{K}^\circ$ are defined by a certain fixed point condition on each these orbits.

If $\mathcal{A}$ is abelian, the fixed point condition is $k\xi = \xi$ and the representation $\varrho^\circ = \sigma^\circ \otimes \chi$ acts on the Hilbert space $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^\sigma \otimes \mathbb{C}$. We note that, if $\mathcal{A}$ is abelian, $\mathcal{A} \simeq \mathbb{R}^n$ under addition and the representations are the characters $\xi_c(a) = \chi_c(a) = e^{ia\cdot c}$ and therefore $\mathbf{H}^\xi \simeq \mathbb{C}$.

If $\mathcal{A}$ is not abelian, the fixed point condition is $k\xi = \frac{1}{s}\rho(k)\xi\rho(k)^{-1}$ and the representation $\varrho^\circ = \sigma^\circ \otimes \rho$ acts on the Hilbert space $\mathbf{H}^{\varrho^\circ} \simeq \mathbf{H}^{\sigma^\circ} \otimes \mathbf{H}^\xi$. ρ is a projective extension of the representation ξ to $\mathcal{G}^\circ$, $\rho(g) : \mathbf{H}^\xi \to \mathbf{H}^\xi$ for $g \in \mathcal{G}^\circ$ with $\rho|_{\mathcal{A}} \simeq \xi$. If $\mathcal{A}$ is abelian, the extension is trivial, $\rho|_{\mathcal{K}} \simeq 1$ and this reduces to the abelian case above.

These representations may be lifted to the algebra. Define $T_e\xi = \xi'$, $T_e\sigma^\circ = \sigma^{\circ\prime}$ and $T_e\varrho^\circ = \varrho^{\circ\prime}$. Then for $A \in \mathbf{a}(\mathcal{A}) \simeq T_e\mathcal{A}$, $Z \in \mathbf{a}(\mathcal{K}^\circ)$ and $W = A + Z \in \mathbf{a}(\mathcal{G}^\circ)$ we have

$$\begin{aligned} &\varrho^{\circ\prime}(W) : \mathbf{H}^{\varrho^\circ} \to \mathbf{H}^{\varrho^\circ} \\ &= \sigma^{\circ\prime}(Z) \oplus \rho'(W) : \mathbf{H}^{\sigma^\circ} \otimes \mathbf{H}^\xi \to \mathbf{H}^{\sigma^\circ} \otimes \mathbf{H}^\xi \\ &: |\psi\rangle \mapsto \left|\tilde{\psi}\right\rangle = \sigma^{\circ\prime}(Z)\,|\varphi\rangle \otimes |\phi\rangle \oplus |\varphi\rangle \otimes \rho'(W)\,|\phi\rangle\,. \end{aligned} \tag{26}$$

The basis of the algebra satisfies the Lie algebra

$$\begin{aligned} [A_\mu, A_\nu] &= c^\lambda_{\mu,\nu} A_\lambda, \\ [Z_\alpha, Z_\beta] &= c^\gamma_{\alpha,\beta} Z_\gamma, \\ [A_\mu, Z_\alpha] &= c^\nu_{\mu,\alpha} A_\nu. \end{aligned} \tag{27}$$

where $\alpha, \beta.. = 1...\dim(\mathcal{K})$ and $\mu, \nu.. = 1,..\dim(\mathcal{A})$. Then, the Hermitian projective extension representation ρ of the generators satisfies the commutation relations

$$\begin{aligned} [\rho'(A_\mu), \rho'(A_\nu)] &= ic^\lambda_{\mu,\nu}\rho'(A_\lambda), \\ [\rho'(Z_\alpha), \rho'(Z_\beta)] &= i\tfrac{1}{s}c^\gamma_{\alpha,\beta}\rho'(Z_\gamma), \\ [\rho'(A_\mu), \rho'(Z_\alpha)] &= i\tfrac{1}{s}c^\nu_{\mu,\alpha}\rho'(A_\nu). \end{aligned} \tag{28}$$

As the $\rho'(Z_\alpha)$ act on the Hilbert space $\mathbf{H}^\xi$, they must be elements of the enveloping algebra $\mathbf{a}(\mathcal{A}) \simeq \mathbf{a}(\mathcal{A}) \oplus \mathbf{a}(\mathcal{A}) \otimes \mathbf{a}(\mathcal{A}) \oplus \ldots$. and therefore

$$\varrho'(Z_\alpha) = d^\mu_\alpha \xi'(A_\mu) + d^{\mu,\nu}_\alpha \xi'(A_\mu)\,\xi'(A_\nu) + \ldots. \tag{29}$$

These may be substituted into the commutation relations above to determine the constants $\{d^\mu_\alpha, d^{\mu,\nu}_\alpha, ...\}$.

We have now characterized the representations ϱ° acting on $\mathbf{H}^{\varrho^\circ}$ and it remains to use the Mackey induction theorem to obtain the representations on the full group $\mathcal{G}$. Clearly, if $\mathcal{G}^\circ \simeq \mathcal{G}$, this implies that $\varrho = \varrho^\circ$ and $\mathbf{H}^{\varrho^\circ} = \mathbf{H}^\varrho$ and the induction is trivial and the representation ϱ of $\mathcal{G}$ is determined.

However, if $\mathcal{G}^\circ$ is a proper closed subgroup of $\mathcal{G}$, then the Mackey induction theorem states that the representations act on the Hilbert space $\mathbf{H}^\varrho \simeq \mathbf{H}^{\sigma^\circ} \otimes \mathbf{L}^2(\mathbb{A}, \mathbf{H}^\xi, \mu)$ with $\mathbb{A} \simeq \mathcal{K}/\mathcal{K}^\circ$ where for $|\phi\rangle \in \mathbf{L}^2(\mathbb{A}, \mathbf{H}^\xi, \mu)$, then $\phi : \mathbb{A} \to \mathbf{H}^\xi : a \mapsto |\phi_a\rangle = \langle a|\phi\rangle$. Also, if $\mathcal{A}$ is abelian, $\mathbf{H}^\xi \simeq \mathbb{C}$ and this reduces to $\phi \in \mathbf{L}^2(\mathbb{A}, \mathbb{C}, \mu)$ with $\phi : \mathbb{A} \to \mathbb{C} : a \mapsto \phi(a)$.

Thus if $|\tilde{\psi}\rangle = \varrho(g)|\psi\rangle$ for $g \in \mathcal{G}$ and $|\psi\rangle, |\tilde{\psi}\rangle \in \mathbf{H}^\varrho$, then $|\tilde{\psi}_{\tilde{a}}\rangle = \langle\tilde{a}|\tilde{\psi}\rangle = \langle\tilde{a}|\varrho(g)|\psi\rangle$ where $|\tilde{\psi}_{\tilde{a}}\rangle \in \mathbf{H}^{\varrho^\circ}$. The induction theorem then states that

$$\left|\tilde{\psi}_{\tilde{a}}\right\rangle = \varrho^\circ(\Theta(\tilde{a})^{-1} g \Theta(a))\,|\psi_a\rangle \text{ with } \quad \tilde{a} = ga.$$

where Θ is the natural section $\Theta : \mathbb{A} \to \mathcal{G} : a \mapsto g = \Theta(a)$. Note that for $g^\circ \in \mathcal{G}^\circ$, $g^\circ a = a = \tilde{a}$ and as $\Theta(a) \in \mathcal{G}^\circ$, this reduces to just an inner automorphism of $\mathcal{G}^\circ$ that defines the equivalence classes of ρ° and this reduces to the expected $|\tilde{\psi}_a\rangle = \varrho^\circ(g^\circ)|\psi_a\rangle$. Putting it together, we have the induced representation theorem where the representations ϱ° of $\mathcal{G}^\circ$ on $\mathbf{H}^{\varrho^\circ}$ are induced onto the representations ϱ of $\mathcal{G}$ on $\mathbf{H}^\varrho$ by

$$\langle a|\,\varrho(g)\,|\psi\rangle = \left|\tilde{\psi}_a\right\rangle = \varrho^\circ(\Theta(a)^{-1} \cdot g \cdot \Theta(g^{-1}a))\left|\psi_{g^{-1}a}\right\rangle.$$

Again, in the abelian case, $|\psi_{g^{-1}a}\rangle \in \mathbf{H}^\xi \simeq \mathbb{C}$ in which case this is written simply as $\psi(g^{-1}a)$

4.2 Mackey representations of the Heisenberg group

The Heisenberg group $\mathcal{H}(n+1) \simeq \mathcal{T}(n+1) \otimes_s \mathcal{T}(n+2)$ is the normal subgroup of the quaplectic group and we therefore review the Mackey representation theory [5],[15] of the Heisenberg group briefly as it is required for the representations of the quaplectic group.

$\mathcal{A} \simeq \mathcal{T}(n+2)$ is the normal subgroup with an algebra spanned by $\{I, Y_a\}$ and $\mathcal{K} \simeq \mathcal{T}(n+1)$ is the homogeneous group with an algebra spanned by $\{X_a\}$. We use χ for the notation for the representation of the normal group $\mathcal{A}$ in this section which, for the translation group, are just the characters

$$\begin{aligned} \chi_{c,u}(h(0,y,\iota))\,|z\rangle &= e^{i\left(\iota\chi'(I) + y^a\chi'(Y_a)\right)}\,|z\rangle \\ &= e^{i(\iota c + y\cdot u)}\,|z\rangle\,, \end{aligned} \tag{30}$$

where $\chi'(I)|z\rangle = c|z\rangle$ and $\chi'(Y_a)|z\rangle = u_a|z\rangle$. $c \in \mathbb{R}$, $u \in \mathbb{R}^{n+1}$ are the Casimir eigenvalues of the translation group and $|z\rangle \in \mathbf{H}^\chi \simeq \mathbb{C}$. The action on the dual $\hat{\mathcal{A}}$ defining the orbits is

$$\begin{aligned} &(h(x,0,0)\chi_{c,u})\,(h(0,y,\iota)) \\ &= \chi_{u,c}\ \left(h(x,0,0)\cdot h(0,x,\iota)\cdot h(x,0,0)^{-1}\right) \\ &= \chi_{u+cy,c}\,(h(0,y,\iota))\,, \end{aligned}$$

for all $h(0,y,\iota) \in \mathcal{T}(n+1)$. Therefore, the fixed point condition is $\chi_{u,c} = \chi_{u+cy,c}$. One solution is $c = 0$ in which case the representation $\chi'(I)|z\rangle = 0$ and hence the representation is a degenerative case equivalent to the representations of $\mathcal{T}(2n)$ that are not considered further. For $c \neq 0$, the fixed point is satisfied only if $u = 0$ and so the little group is trivial, $\mathcal{K}^\circ \simeq e$. Consequently, $\varrho^\circ = \chi$. The homogeneous space is $\mathbb{A} \simeq \mathcal{K}/\mathcal{K}^\circ \simeq \mathcal{T}(n+1)/e \simeq \mathbb{R}^{n+1}$. The Hilbert space is therefore

$$\mathbf{H}^\varrho \simeq \mathbf{L}^2(\mathbb{A}, \mathbf{H}^\xi, \mu) \simeq \mathbf{L}^2(\mathbb{R}^{n+1}, \mathbb{C}, \mu) \tag{31}$$

as expected. I is the only Casimir invariant operator and its eigenvalues label the representations. Points $a_x \in \mathbb{A}$ are labeled by $h(x,0,0) \in \mathcal{K}$ with $\Theta(a_x) = h(x,0,0)$. $h(x,y,\iota)a_{\tilde{x}} = a_{\tilde{x}+x}$ and therefore the Mackey induction theorem yields

$$\begin{aligned} \tilde{\psi}(a_y) &= \langle a_x|\,\varrho(h(\tilde{x},\tilde{y},\iota))\,|\psi\rangle \\ &= \varrho^\circ(h(0,\tilde{y},\iota - \tilde{y}\cdot x))\psi(a_{x-\tilde{x}}). \end{aligned}$$

where a calculation showing that

$$h(0,\tilde{y},\iota-\tilde{y}\cdot x)=h(\tilde{x},\tilde{y},\iota)\cdot\Theta(h(\tilde{x},\tilde{y},\iota)^{-1}a_x)$$

and $a_{x-\tilde{x}}=h(\tilde{x},\tilde{y},\iota)^{-1}a_x$ has been used. Using the bijection $a_x\leftrightarrow x$ and the definition of $\varrho^{\circ}=\chi$ in (30) with $u=0$ this gives

$$\tilde{\psi}(x)=e^{i(\iota-\tilde{y}\cdot x)c}\psi(x-\tilde{x})=e^{i\iota c}e^{-ic\tilde{y}\cdot x}e^{i\ \ \tilde{x}\cdot i\frac{\partial}{\partial x}}\psi(x).\quad(32)$$

It follows that the representation of the algebra is

$$\begin{array}{l}\langle x|\,\varrho'(I)\,|\psi\rangle=c\,|\psi\rangle\,,\\ \langle x|\,\varrho'(X_a)\,|\psi\rangle=c\ \ x^a\,|\psi\rangle\,,\\ \langle x|\,\varrho'(Y_a)\,|\psi\rangle=i\frac{\partial}{\partial x}\,|\psi\rangle\,.\end{array}\qquad(33)$$

Thus, it is clear that the usual basic representations of quantum mechanical position and momentum Hermitian operators and their associated Hilbert space are directly computed using the Mackey representation theory of the Heisenberg group.

4.3 Mackey representations of the quaplectic group

The normal subgroup of the quaplectic group is the Heisenberg group and so the Mackey case for nonabelian normal subgroups applies. As the Heisenberg is now the normal subgroup, we denote the representations ϱ from the previous section given in (32) in this section by ξ. These representations ξ_c of $\mathcal{H}(n+1)$ are labeled by the eigenvalues c of the Casimir invariant operator I. The case $c=0$ corresponds to the degenerate abelian case, that we do not consider further here.

The general nonabelian case corresponds to $c\in\mathbb{R}\backslash\{0\}$. This requires the definition of a projective extension ρ of ξ with the property that $\rho|_{\mathcal{H}(n+1)}=\xi$. The action of $\Upsilon\in\mathcal{U}(1,n)$ on the representations ξ of the normal subgroup is

$$\begin{array}{l}(\Upsilon\xi_c)\,(g(0,z,\iota))=(\Upsilon\rho)\,(g(0,z,\iota))\\ =\rho(g(\Upsilon,0,0)\cdot g(0,z,\iota)\cdot g(\Upsilon,0,0)^{-1})\\ =\rho(\Upsilon)\cdot\xi_c(g(0,z,\iota))\cdot\rho(\Upsilon^{-1})\end{array}$$

for all $g(0,z,\iota)\in\mathcal{H}(n+1)$. Consequently, the fixed point condition for the nonabelian case

$$\Upsilon\cdot\xi_c=\rho(\Upsilon)\cdot\xi_c\cdot\rho(\Upsilon)^{-1}$$

is identically satisfied for all $\Upsilon\in\mathcal{U}(1,n)$. Consequently, the little group is $\mathcal{K}^{\circ}\simeq\mathcal{U}(1,n)\simeq\mathcal{K}$ [16].

An immediate consequence of this is that the Mackey induction is trivial and the Hilbert space of the representations is given simply by the direct product of the Hilbert space of the representations of the little group and the Hilbert space of the representations of the Heisenberg group, $\mathbf{H}^{\varrho}=\mathbf{H}^{\sigma}\otimes\mathbf{H}^{\xi}$. As the Hilbert space $\mathbf{H}^{\xi}$ of the representations ξ of $\mathcal{H}(1,n)$ have been determined in (31), this is

$$\mathbf{H}^{\varrho}=\mathbf{H}^{\sigma}\otimes\mathbf{L}^2(\mathbb{R}^{n+1},\mathbb{C},\mu).$$

$\mathbf{H}^{\sigma}$ is generally a countably infinite vector space as the unitary irreducible representations of $\mathcal{U}(1,n)$ are generally infinite dimensional. This is remarkably different from the Poincaré case where the Hilbert space was over the n dimensional null and time-like hyperboloid surfaces in $\mathbb{R}^{n+1}$. The complete set of irreducible representations in that case is required to foliate $\mathbb{R}^{n+1}$ Here a single representation covers the entire space $\mathbb{R}^{n+1}$ and so the particle states are not constrained by the representation to a hypersurface.

The representations ϱ are the direct product of σ and ρ given by

$$\begin{array}{l}\varrho(g(\Upsilon,z,\iota)):\mathbf{H}^{\varrho}\to\mathbf{H}^{\varrho}:|\psi\rangle\simeq|\varphi\rangle\otimes|\phi\rangle\mapsto\left|\tilde{\psi}\right\rangle\\ \quad=\varrho(g(\Upsilon,z,\iota))\,|\psi\rangle\simeq\sigma(g(\Upsilon,0,0))\,|\varphi\rangle\otimes\rho(g(\Upsilon,z,\iota))\,|\phi\rangle\,.\end{array}$$

where, as above, $|\psi\rangle\in\mathbf{H}^{\varrho}$, $|\varphi\rangle\in\mathbf{H}^{\sigma}$, $|\phi\rangle\in\mathbf{H}^{\xi}\simeq\mathbf{L}^2(\mathbb{R}^{n+1},\mathbb{C},\mu)$.

The full representations ϱ may be lifted to the algebra ϱ' to act on a basis $\{A_b^{\pm},Z_{a,b}\}$ of $\mathbf{a}(\mathcal{Q}(1,n))$. Using (12) and noting that as the representation is Hermitian, the appropriate factors of i must be inserted,

$$\begin{array}{l}\left[\varrho'(A_a^{+}),\varrho'(A_b^{-})\right]=\eta_{a,b}\varrho'(I),\\ \left[\varrho'(Z_{a,b}),\varrho'(A_c^{\pm})\right]=\mp\eta_{a,c}\varrho'(A_b^{\pm}),\\ \left[\varrho'(Z_{a,b}),\varrho'(Z_{c,d})\right]=\varrho'(Z_{c,b})\eta_{a,d}-\varrho'(Z_{a,d})\eta_{b,c}.\end{array}\qquad(34)$$

The representations ρ may also be lifted to the algebra ρ' to act on a basis $Z_{a,b}$ of $\mathbf{a}(\mathcal{U}(1,n))$. Using (28), the representations of the algebra are

$$\begin{array}{l}\left[\xi'(A_a^{-}),\xi'(A_b^{+})\right]=\eta_{a,b}\xi'(I),\\ \left[\rho'(Z_{a,b}),\xi'(A_c^{\pm})\right]=\pm\frac{1}{s}\eta_{a,c}\xi'(A_b^{\pm}),\\ \left[\rho'(Z_{a,b}),\rho'(Z_{a,b})\right]=\frac{1}{s}\left(\eta_{b,c}\rho'(Z_{a,d})-\eta_{a,d}\rho'(Z_{c,b})\right).\end{array}$$

These elements must be in the enveloping algebra $\mathbf{e}(\mathcal{U}(1,n))$ and therefore may be expressed, using (29), as

$$\rho'(Z_{a,b})=d_{\pm}^{a}\xi'(A_a^{\pm})+d_{\pm,\pm}^{a,b}\xi'(A_a^{\pm})\xi'(A_b^{\pm})+...$$

Substituting into the above commutation relations gives

$$\rho'(Z_{a,b})=\frac{1}{s}\xi'(A_a^{+})\xi'(A_b^{-}).\qquad(35)$$

The representation of the algebra is given by

$$\begin{array}{l}\varrho'(I)\,|\psi\rangle=|\varphi\rangle\otimes\xi'(I)\,|\phi\rangle\,,\\ \varrho'(A_a^{\pm})\,|\psi\rangle=|\varphi\rangle\otimes\xi'(A_a^{\pm})\,|\phi\rangle\,,\\ \varrho'(Z_{a,b})\,|\psi\rangle=\sigma'(Z_{a,b})\,|\varphi\rangle\otimes|\phi\rangle\oplus|\varphi\rangle\otimes\rho'(Z_{a,b})\,|\phi\rangle\end{array}$$

where $\rho'(Z_{a,b})$ is defined in (36). The matrix elements of $\xi'(A_a^{\pm})$ and $\rho'(Z_{a,b})$ with respect to a basis $|x\rangle$ are given by

$$\begin{array}{l}\langle x|\,\xi'(A_a^{\pm})\,|\psi\rangle=\frac{1}{\sqrt{2}}\left(x\pm\frac{\partial}{\partial x^a}\right),\\ \langle x|\,\rho'(Z_{a,b})\,|\psi\rangle=\frac{1}{2}(x^a+\frac{\partial}{\partial x^a})(x^a-\frac{\partial}{\partial x^a}).\end{array}\qquad(36)$$

The matrix elements for the $\sigma'(Z_{a,b})$ are countably infinite matrices that will be defined further shortly,

$$(\Sigma_{a,b})_{\tilde{M},M}=\left\langle\tilde{M}\right|\sigma'(Z_{a,b})\,|M\rangle$$

where $\tilde{M}, M$ take values in a countably infinite set. Then, the expression for the unitary irreducible representations of the group may be written explicitly as

$$\begin{aligned}&\left\langle \tilde{M}, x \right| \varrho(g(\Upsilon(\zeta), z, \iota)) \left|\psi\right\rangle = \\ &e^{i\zeta^{a,b}(\Sigma_{a,b})_{\tilde{M},M}} \otimes e^{i(\iota I + \frac{1}{\sqrt{2}} z^a_\pm (x^a \pm \frac{\partial}{\partial x^a}))} \times \\ &e^{i\frac{1}{2}\zeta^{a,b}(x^a + \frac{\partial}{\partial x^a})(x^b - \frac{\partial}{\partial x^b})} \psi_M(x)\end{aligned}$$

with $z^a_+ = z^a$ and $z^a_- = \overline{z}^a$. This completes the characterization of the unitary irreducible representations of the quaplectic group and algebra given by the Mackey theory.

We now characterize the representations of the algebra and the Hilbert spaces on which they act in further detail. Following this, the form of the general field equations for the quaplectic group are given and specific cases studied.

4.4 The representations ρ' of the algebra of $\mathcal{Q}(1,n)$.

The representations ρ' of the algebra of the full group $\mathcal{Q}(1,n)$ act on the Hilbert space $\mathbf{H}^\xi \simeq \mathbf{L}^2(\mathbb{R}^{n+1}, \mathbb{C})$ of the representations ξ of the normal subgroup $\mathcal{H}(1,n)$. The representations ρ' are the extension to the full set of generators $\{Z_{a,b}, A^\pm_a, I\}$, $a, b, ... = 0, 1, ...n$ of $\mathcal{Q}(1,n)$ where $\rho'(A^\pm_a) = \xi'(A^\pm_a)$ and $\rho'(I) = \xi'(I)$ and $\rho'(Z_{a,b})$ is given by equation (35).

A coherent basis for the Hilbert space $\mathbf{H}^\xi$ may be defined by the complete set of orthonormal states $|\eta_{k_0,k_1,...k_n}\rangle$ [17]

$$\langle x_a|\eta_{k_a}\rangle = \eta_{k_a}(x_a) = \frac{1}{\sqrt{\pi 2^{k_a} k_a!}} e^{-\frac{1}{2}(x_a)^2} H_{k_a}(x_a), \quad (37)$$

with $k_a \in \mathbb{Z}^+$, $x_a \in \mathbb{R}$ and $|\eta_K\rangle = |\eta_{k_0,k_1,...k_n}\rangle$ with $K = (k_0, k_1, ...k_n)$.
Defining I_a now to be an $n+1$ tuple of 0's with a 1 at the ath position, $I_a = (0, 0, , , 1, ...0)$, the action of the representation on the basis is

$$\begin{aligned}\xi'(A^-_0)\left|\eta_K\right\rangle &= \sqrt{k_0+1}\left|\eta_{K+I_0}\right\rangle, \\ \xi'(A^+_0)\left|\eta_K\right\rangle &= \sqrt{k_0}\left|\eta_{K-I_0}\right\rangle, \\ \xi'(A^+_i)\left|\eta_K\right\rangle &= \sqrt{k_i+1}\left|\eta_{K+I_i}\right\rangle, \\ \xi'(A^-_i)\left|\eta_K\right\rangle &= \sqrt{k_i}\left|\eta_{K-I_i}\right\rangle,\end{aligned} \quad (38)$$

where as always $i, j = 1, 2, ..n$. It follows directly that

$$\begin{aligned}\rho'(Z_{i,0})\left|\eta_K\right\rangle &= \tfrac{1}{s}\sqrt{(k_0+1)(k_i+1)}\left|\eta_{K+I_i+I_0}\right\rangle, \\ \rho'(Z_{0,i})\left|\eta_K\right\rangle &= \tfrac{1}{s}\sqrt{k_i k_0}\left|\eta_{K-I_i-I_0}\right\rangle, \\ \rho'(Z_{j,i})\left|\eta_K\right\rangle &= \tfrac{1}{s}\sqrt{(k_j+1)k_i}\left|\eta_{K-I_i+I_j}\right\rangle, i \neq j, \\ \rho'(Z_{a,a})\left|\eta_K\right\rangle &= \tfrac{1}{s}k_a\left|\eta_K\right\rangle.\end{aligned} \quad (39)$$

The little group may be factored as $\mathcal{U}(1,n) = \mathcal{U}(1) \otimes \mathcal{SU}(1,n)$. Note that the generator U of the algebra of $\mathcal{U}(1)$ is defined by $U = \eta^{a,b} Z_{a,b}$ and so the representation is

$$\rho'(U)\left|\eta_K\right\rangle = (-k_0 + k_1 + k_2 + ...k_n)\left|\eta_K\right\rangle = k\left|\eta_K\right\rangle \quad (40)$$

with $k = -k_0 + \sum_{i=1}^{n} k_i$ where again $k_a \in \mathbb{Z} \geq 0$. The Hilbert space $\mathbf{H}^\xi$ may be written as a direct sum $\mathbf{H}^\xi = \bigoplus_{k=1}^{\infty} \mathbf{H}^\xi_k$ of subspaces $\mathbf{H}^\xi_k$ that are invariant under the ρ'representations of the generators $Z_{a,b}$ of $\boldsymbol{a}(\mathcal{U}(1,n))$

$$\rho'(Z_{a,b}) : \mathbf{H}^\xi_k \to \mathbf{H}^\xi_k : |\tilde{\eta}_K\rangle \mapsto \rho'(Z_{a,b})\left|\eta_K\right\rangle. \quad (41)$$

The representations of the generators $\rho'(A^\pm_a)$ of $\boldsymbol{a}(\mathcal{H}(1,n))$ cause transitions between $\mathbf{H}^\xi_k$ subspaces with different k.

$$\rho'(A^\pm_i) : \mathbf{H}^\xi_k \to \mathbf{H}^\xi_{k\pm1}, \quad \rho'(A^\pm_0) : \mathbf{H}^\xi_k \to \mathbf{H}^\xi_{k\mp1}$$

As $\mathcal{U}(1,n)$ is not compact, each $\mathbf{H}^\xi_k$ is countably infinite dimensional due to the indefinite signature. Define $l = \sum_{i=1}^{n} k_i$, and the l label the finite dimensional $\mathcal{U}(n)$ invariant subspaces $\mathbf{H}^\xi_l \simeq \mathbf{V}^l$. Thus, the $\mathbf{H}^\xi_k$ may, in turn, be written as an infinite direct sum of the finite dimensional Hilbert spaces $\mathbf{H}^\xi_k = \bigoplus_{l=k}^{\infty} \mathbf{V}^l$.

4.5 The representations σ' of the algebra of $\mathcal{U}(1,n)$.

The next task is to give the explicit form of the representations $\sigma'(Z_{a,b})$ of the algebra of $\mathcal{U}(1,n)$. This problem has been solved both for the representation of the algebra and also for the group in [18],[19],[20]. These results are briefly summarized in the notation of this paper as follows.

A convenient basis of $\mathbf{H}^\sigma$ is the Gel'fand [21] basis $|M\rangle$ that is derived from the subgroup chain $\mathcal{U}(1,n) \supset \mathcal{U}(n) \supset \mathcal{U}(n-1) \supset ... \supset \mathcal{U}(1)$. This basis may be written explicitly as

$$|M\rangle = \left| \begin{array}{ccccccc} m_{1,n+1} & & m_{2,n+1} \cdots & & m_{n,n+1} & & m_{n+1,n+1} \\ & m_{1,n} & & \cdots & & m_{n,n} & \\ & & & \cdots & & & \\ & & & m_{1,1} & & & \end{array} \right\rangle$$

where the $m_{i,j}$ satisfy the inequalities

$$m_{i,j} \geq m_{i+1,j}, \qquad m_{i,j} \geq m_{i,j-1} \geq m_{i+1,j} \quad .$$

If you put the above inequalities into the triangular form above, a simple pattern emerges. The integers $m_{i,n+1}$ label the irreducible representations and the eigenvalues in this basis of the Casimir operators of $\mathcal{U}(1,n)$ are given in terms of these quantities as is described shortly. The remainder of the $m_{i,j}$ label states within the irreducible representation.

The full set of matrix elements of the representation

$$(\Sigma_{a,b})_{\tilde{M},M} = \left\langle \tilde{M} \right| \sigma'(Z_{a,b}) \left|M\right\rangle \quad (42)$$

are determined by the matrix elements of $\sigma'(Z_{a,a})$, $\sigma'(Z_{a,a+1})$ and $\sigma'(Z_{a+1,a})$. The remainder may be computed directly from these using the Lie algebra relations.

We need to convert between the indices $a, b = 0, 1...n$ used throughout the paper and the $j, k.. = 1, ...n+1$ that label the Gel'fand basis. We therefore defined $\hat{k} = k$ for

$k = 1,...n$ and $\hat{k} = 0$ for $k = n+1$. Then, the matrix elements, as given in [18],[19] are

$$\begin{aligned}
\sigma'(Z_{\hat{k},\hat{k}})\,|M\rangle &= \epsilon_{k,k}(\sum_{i=1}^{k} m_{k,i} - \sum_{i=1}^{k-1} m_{k-1,i})\,|M\rangle\,, \\
\sigma'(Z_{\hat{k},\hat{k+1}})\,|M\rangle &= \epsilon_{k,k+1}\sum_{i=1}^{k} s_{k,i}(M)\,|M+I_{k,i}\rangle\,, \\
\sigma'(Z_{\hat{k+1},\hat{k}})\,|M\rangle &= \epsilon_{k+1,k}\sum_{i=1}^{k} s_{k,i}(M-I_{k,i})\,|M-I_{k,i}\rangle\,,
\end{aligned} \tag{43}$$

where $\epsilon_{k,j} = 1$ for $j,k = 1,..n$, -1 for $j = k = n+1$ and i otherwise. $|I_{l,k}\rangle$ is the state where all the $m_{i,j} = 0$ unless $i = l$ and $k = j$ in which case $m_{l,k} = 1$. The $s_{k,j}$ are the functions

$$s_{k,j}(M) \doteq i\frac{\prod_{i=1}^{k+1} r_{i,j}(k+1,0) \quad \prod_{i=1}^{k-1} r_{i,j}(k-1,1)}{\prod_{i=1,i\neq j}^{k-1} r_{i,j}(k,1)r_{i,j}(k,0)}. \tag{44}$$

where

$$r_{i,j}(k,a) = \sqrt{m_{k,i} - m_{k,j} - i + j - a}$$

Now, for the compact case where the indices $a,b = 0,1,..n$ are restricted to the case $i,j = 1,..n$, the representations are finite dimensional. In the non-compact case, the indices $m_{i,k}$ labeling the states are not generally finite dimensional. However, representations of $\mathcal{U}(1,n)$ may be decomposed into infinite direct sums of the finite dimensional $\mathcal{U}(n)$ representations. In this case, $m_{i,n}$ label the irreducible finite dimensional representations of $\mathcal{U}(n)$ and, again, an infinite number of these finite dimensional irreducible representations of $\mathcal{U}(n)$ appear in each irreducible representation of $\mathcal{U}(1,n)$.

A simple direct computation shows that the eigenvalue of the representation of the first Casimir invariant U is given by

$$\sigma'(U)\,|\varphi\rangle = \eta^{a,b}\sigma'(Z_{a,b})\,|\varphi\rangle = d_1\,|\varphi\rangle$$

where

$$\begin{aligned}
d_1 &= \eta^{a,b}(\Sigma_{a,b})_{M,M} = \sum_{a=0}^{n} \eta^{a,a} m_{\hat{a},n+1} \\
&= m_{1,n+1} + m_{2,n+1}.....m_{n,n+1} - m_{n+1,n+1}
\end{aligned}. \tag{45}$$

Similarly, higher order Casimir eigenvalues d_α may similarly be computed in terms of the $\{m_{k,n+1}\}$ with $k = 1,..n+1$.

5 Field equations of the quaplectic group

5.1 The general Casimir field equations

The field equations are defined in (2) to be

$$\varrho'(C_\alpha)\,|\psi\rangle = c_\alpha\,|\psi\rangle \text{ with } |\psi\rangle \in \mathbf{H}^{\varrho} \quad, \alpha = 1,2...N_c \tag{46}$$

where for $\mathcal{Q}(1,n)$ $N_c = n+2$ and, in addition to $C_1 = I$, the C_α are defined in terms of the $W_{a,b}$ given in (14) ,

$$C_{2\beta} = \eta^{a_1,a_{2\beta}}...\eta^{a_{2\beta-2},a_{2\beta-1}} W_{a_1,a_2}...W_{a_{2\beta-1},a_{2\beta}}, \tag{47}$$

with $\beta = 1,..n+1$. The representation $\varrho'(W_{a,b})$ of these generators is

$$\varrho'(W_{a,b})\,|\psi\rangle = c\sigma'(Z_{a,b})\,|\varphi\rangle\otimes|\phi\rangle \oplus |\varphi\rangle\otimes a\xi'(A_a^+)\xi'(A_b^-)\,|\phi\rangle \tag{48}$$

where $a = 1 - \frac{c}{s}$. Again, $c = c_1$ is the eigenvalue of the representation,

$$\varrho'(I)\,|\psi\rangle = c\,|\psi\rangle\,.$$

The representation of the general Casimir invariants $\varrho'(C_\alpha)$ involve products of the $\varrho'(W_{a,b})$ and therefore $\xi'(A_a^{\pm})$. As the matrix elements of the representations $\xi'(A_a^{\pm})$ are given in (36) in terms of the differential operators, this appears to imply that the field equations will be higher order differential equations. However, the Lie algebra may be used to rearrange the terms defining the Casimir invariant operators so that it may be established that the invariants will result in no more than second order differential equations. Using the algebra for the representation of the generators (34), the expression for the representation of the Casimir invariant operator $\varrho'(C_4)$ given by substituting (48) and (47) into (46) may be rearranged into the form

$$\begin{aligned}
&\eta^{a,c}\eta^{b,d}\xi'(A_a^-)\sigma'(Z_{c,d})\xi'(A_b^+)\,|\psi\rangle \\
&= \sum_{\alpha=1}^{2}\sum_{\kappa=0}^{2-\alpha+1} f_{\kappa,\alpha}^2(n+1,a,c_\gamma)\,(\sigma'(D_\alpha))^\kappa\,|\psi\rangle\,.
\end{aligned} \tag{49}$$

where the D_α are the Casimir invariants of the unitary group given in (17), $D_1 = U$ and $D_2 = \eta^{a,b}\eta^{c,d}Z_{a,d}Z_{b,c}$ and the co-efficient functions depend on the dimension of the space, the constant $a = 1 - \frac{c}{s}$ and the Casimir eigenvalues c_γ, $\gamma = 1,2,4$, given in (46) that are constants for each of the irreducible representations of the quaplectic group. The explicit form of these co-efficient functions for the forth order eigenvalue equation are

$$\begin{aligned}
&f_{0,1}^2(n,a,c_\gamma) = \tfrac{1}{2}\left(cn(1+n) - \left(\tfrac{n+1}{a} + n - 1\right)c_2 + \tfrac{c_2^2+c_4}{ac}\right), \\
&f_{1,1}^2(n,a,c_\gamma) = \tfrac{1}{2a}\left((1+a)\,c(1+n) - 2c_2\right), \\
&f_{2,1}^2(n,a,c_\gamma) = \tfrac{1}{2a}c, \\
&f_{2,2}^2(n,a,c_\gamma) = \tfrac{1}{2a}c.
\end{aligned} \tag{50}$$

The corresponding functions for the representation of the Casimir invariant $\varrho'(C_2)$ (15) are simply

$$f_{0,1}^1(n,a,c_\gamma) = c_2/a, \quad f_{1,1}^1(n,a,c_\gamma) = c/a \tag{51}$$

Note that this equation is a second order equation in the operators $\{A_a^{\pm}\}$. Similar expressions may be obtained for the remaining eigenvalue equations (46) for $\beta = 1,2,..n+1$ and it may be verified by direct computation that, (at least up to C_8), these equations are also second order equations in terms of the operators $\{A_a^{\pm}\}$. In fact, the equation for $C_{2\beta}$ with $\beta = 1,..n+1$ is given

by

$$\begin{array}{l}\eta^{a_1,a_2}...\eta^{a_{2\beta-1},a_{2\beta}}\xi'(A^-_{a_1})\sigma'(Z_{a_2,a_3})\times ..\\ \quad .... \times \sigma'(Z_{a_{2\beta-2},a_{2\beta-1}})\xi'(A^+_{a_{2\beta}})\,|\psi\rangle\\ = \sum\limits_{\alpha=1}^{\beta}\sum\limits_{\kappa=0}^{\beta-\alpha+1} f^{\beta}_{\kappa,\alpha}(n+1,a,c_\gamma)\,(\sigma'(D_\alpha))^\kappa\,|\psi\rangle\end{array} \tag{52}$$

with $\gamma = 1, 2, ...2\beta$.

Furthermore, as from (19) the D_α and the $C_{2\beta}$ commute, the state $|\psi\rangle$ may be taken to be eigenfunctions of the representations of D_α, $\sigma'(D_\alpha)|\psi\rangle = d_\alpha|\psi\rangle$. The d_α are not constants on the irreducible representation but rather label states within the representation. The matrix elements of the representations σ' in the $|M\rangle$ basis are defined in terms of the $\Sigma_{a,b}$ by (42-44) and we use this to define

$$\Sigma^{\beta}_{b,c} = \eta^{a_2,a_3}...\eta^{a_{2\beta-2},a_{2\beta-1}}\Sigma_{b,a_2}..\Sigma_{a_{2\beta-1},c}.$$

Finally, using (36) to give the matrix elements of $\xi'(A^{\pm}_a)$ with respect to the $|x\rangle$ basis, the general field equations may be written as the eigenvalue equation

$$\begin{array}{l}\Sigma^{\beta}_{b,c},_{\tilde{M},M}\left(x^b - \frac{\partial}{\partial x^b}\right)\left(x^c + \frac{\partial}{\partial x^c}\right)\psi_M(x)\\ = \hat{f}^{\beta}(n,a,c_\gamma,d_\alpha)\psi_{\tilde{M}}(x)\end{array} \tag{53}$$

with $c = c_1$ and the functions $\hat{f}^{\beta}$ given by

$$\hat{f}^{\beta}(n,a,c_\gamma,d_\alpha) = \sum_{\alpha=1}^{\beta}\sum_{\kappa=0}^{\beta-\alpha+1} f^{\beta}_{\kappa,\alpha}(n+1,a,c_\gamma)d^{\kappa}_{\alpha}$$

where $\alpha = 1, ...\beta$ and $\gamma = 1, 2, ...2\beta$. Thus, the field equations are a set of simultaneous second order differential equations. The $\Sigma^{\beta}_{b,c}$ are countably infinite dimensional matrices defined in terms of the $m_{i,j}$, $i, j \leq n$ and the d_α are defined in terms of the $m_{i,n+1}$ label the various σ irreducible representations of $\mathcal{U}(1,n)$ that appear in each irreducible representation of $\mathcal{Q}(1,n)$. The c_γ label the irreducible representations of $\mathcal{Q}(1,n)$.

5.2 The relativistic harmonic oscillator field equation

In the Poincaré case, different field equations result from different representations of the little group. The Klein-Gordon equation resulted from the trivial representation. In the quaplectic group case, the trivial representation $\sigma'(Z_{a,b}) = 0$ results in field equations that are not invariant under the generators of the Heisenberg group.

Using the factorization $\mathcal{U}(1,n) \simeq \mathcal{U}(1)\otimes\mathcal{SU}(1,n)$, consider the a representation where the representation $\sigma'(U)$ of $\mathcal{U}(1)$ generator is nontrivial but where the $\sigma'(\hat{Z}_{a,b})$ of $\mathcal{SU}(1,n)$ are trivial, $\sigma'(\hat{Z}_{a,b}) = 0$. The representations of $\mathcal{Q}(1,n)$ therefore degenerate to the representations of the oscillator group $\mathcal{O}s(n+1) \simeq \mathcal{U}(1)\otimes_s\mathcal{H}(n+1)$ that has only two independent Casimir invariant operators. $C_1 = I$ is trivial and the quadratic field equation (54), where the $f^{\beta}_{\kappa,\alpha}$ for $\beta = 1$ is given in (51), is just

$$\eta^{a,b}(x^b - \frac{\partial}{\partial x^b})\left(x^a + \frac{\partial}{\partial x^a}\right) = \frac{c_2 + c\ \ d_1}{a}\psi(x).$$

This may be rearranged to the familiar equation for the relativistic oscillator

$$\left(\eta^{a,b}(x^a x^b - \frac{\partial^2}{\partial x^a \partial x^b}) - \frac{cd_1 - c_2}{a} - (n-1)\right)\psi(x) = 0.$$

This has solutions such that $\lim\limits_{|x|\to\infty}\psi(x) = 0$ only if

$$\frac{1}{a}(c\ \ d_1 - c_2) + (n-1) = 2k + n - 1$$

for all $k \in \mathbb{Z}^+$ and gives the result that $k \equiv d_1$ and $c = 2a$ with $c_2 = 0$. Using the definition of $a = 1 - \frac{c}{s}$, this results in

$$a = \frac{s}{2\,(s-1)},\quad c_1 = c = \frac{s}{(s-1)}.$$

5.3 Field equations in coherent basis

The general field equations can be cast into a simpler form by considering a coherent basis $|\eta_K, M\rangle$ instead of a coordinate basis $|x, M\rangle$. The matrix elements

$$\Xi^{a,b}_{\tilde{K},K} = \langle\eta_{\tilde{K}}|\,\xi'(A^+_a)\xi'(A^-_b)\,|\eta_K\rangle = s\,\langle\eta_{\tilde{K}}|\,\rho'(Z_{a,b})\,|\eta_K\rangle$$

are defined by (4.38-39). The field general field equation is then

$$\Sigma^{\beta}_{a,b},_{\tilde{M},M}\,\Xi^{a,b}_{\tilde{K},K}\psi^k_{M,K} = \hat{f}^{\beta}(n,a,c_\gamma,d_\alpha)\psi^k_{\tilde{M},\tilde{K}}. \tag{54}$$

The Hilbert space $\mathbf{H}^{\varrho}$ is the direct product of countably infinite vector spaces $\mathbf{H}^{\varrho} = \mathbf{H}^{\sigma}\otimes\mathbf{H}^{\xi}$. As noted in (40-41) the Hilbert space decomposes into the direct sum of subspaces $\mathbf{H}^{\xi}_k$ invariant under $\Xi^{a,b}$ and hence the label k on the states that is invariant. Define $\Xi^{k,a,b}$ to be $\Xi^{a,b}$ restricted to $\mathbf{H}^{\xi}_k$. Then, if we consider for a moment the field equations of $\mathcal{Q}(n)$ by restricting the indices $a, b = 0, 1, ..n$ to $i, j = 1, ..n$, the field equations become

$$\Sigma^{\beta}_{i,j},_{\tilde{M},M}\,\Xi^{k,i,j}_{\tilde{K},K}\psi^k_{M,K} = \hat{f}^{\beta}(n,a,c_\gamma,d_\alpha)\psi^k_{\tilde{M},\tilde{K}}. \tag{55}$$

Now, for each k, the $\Xi^{k,i,j}$ are finite dimensional matrices of dimension k with $\mathbf{H}^{\xi}_k \simeq \mathbf{V}^k$. Likewise, as the representation is now compact, the $\Sigma_{i,j}$ and hence $\Sigma^{\beta}_{i,j}$ are finite integer dimensional with dimension and so $\mathbf{H}^{\sigma} \simeq \mathbf{V}^{\dim\sigma}$. All of the quantities are defined and it reduces to a finite matrix eigenvalue problem that is solvable as both the $\Xi^{k,i,j}$ and $\Sigma_{i,j}$ are Hermitian. These give rise to a set of spinning harmonic oscillators as will be discussed in a subsequent paper.

In the non-compact case of general interest, the matrices are countably infinite. Never-the-less, these comments give reason to believe that the solution of these field equations are tractable.

6 Discussion

We were led to the quaplectic group from the very basic goal of obtaining a dynamical group that encompassed both the Poincaré group and the Heisenberg group. One new physical assumption, the Born orthogonal metric hypothesis, was adopted. There are no other essentially new physical assumptions in this paper.

The dynamical group framework has proven its effectiveness in the Poincaré case. We correspondingly obtained the unitary representations of the quaplectic group and the associated field equations that are the eigenvalue equations for the Hermitian representations of the Casimir invariants.

The Poincaré group reduces in the limit of $c \to \infty$ to the Galilei group of nonrelativistic mechanics. The Galilei group acting on the position-time space leaves the time subspace invariant. Thus, in the Galilei case we can speak of absolute time. The Poincaré group does not have this invariant subspace and consequently time is relative to the observer. The Poincaré group *mixes* the position and time degrees of freedom.

The same phenomena occurs with the quaplectic group. The quaplectic group acts on a nonabelian time-position-momentum-energy space. In a companion paper in preparation [22], we show that in the limit $b \to \infty$, the quaplectic group contracts effectively to the Poincaré group. This contracted group, acting on the full nonabelian space, leaves invariant the position-time subspace. Thus, in this limit there is the concept of absolute position-time space, or as it is more usually stated, *space-time*. However, under the full quaplectic group, this is no longer the case. For strongly interacting states, *space-time* is relative to the observer and in general, of the degrees of freedom of the nonabelian space *mix*. That is, space-time can be transformed into energy and momentum and vice versa.

The field equations of the quaplectic group may be explicitly determined. The simplest equation in the set is the relativistic oscillator that is the counterpart in this theory of the Klein-Gordon equation of the Poincaré case. In the general case, the Hilbert space $\mathbf{H}^{\sigma}$ of the internal degrees of freedom corresponding to the little group $\mathcal{U}(1,n)$ is generally infinite dimensional. This is unlike the compact group where the little groups of physical interest are compact and finite dimensional.

The Hilbert space $\mathbf{H}^{\xi}$ decomposes into spaces $\mathbf{H}^{\xi}_{k}$ that are invariant under the ρ representation of the generators of the little group. This ρ representation arises because the normal subgroup $\mathcal{H}(n+1)$ is non abelian and has no counterpart in the Poincaré case.

How do we reconcile these infinite dimensional internal degrees of freedom? The $\mathcal{U}(1,n)$ representations may be decomposed into an infinite ladder of finite dimensional $\mathcal{U}(n)$ representations. Preliminary results indicate that under the group contractions, these ladders *break* so that each finite dimensional representation that is a rung of the ladder of representations defining the infinite dimensional irreducible $\mathcal{U}(1,n)$ representation becomes a (finite) irreducible representation of the contracted group. This requires further investigation but we know that particle states seem to appear in ladders although we have apparently probed only the first three rungs in the interactions that are accessible.

The theory bounds relative rates of change of momentum, force, in addition to the usual rates of change of position, velocity. This is embodied by the four distinct Poincaré subgroups with representations of translation generators represented by the quad

$$\begin{array}{ccc} \varrho'(T) & \leftrightarrow & \varrho'(Q_i) \\ \updownarrow & & \updownarrow \\ \varrho'(P_i) & \leftrightarrow & \varrho'(E) \end{array}$$

The Lorentz group of two of these Poincaré groups is the transformation of the usual (velocity) special relativity and the Lorentz group of the remaining two Poincaré groups is the reciprocally conjugate (force) special relativity. The representation of the translation generators of only one of the four Poincaré subgroups can be simultaneously diagonalized, and therefore observed, in this nonabelian space. These are the 4 faces of the quad.

Should the quaplectic dynamical group have experimental basis, many of the Poincaré group physical concepts become approximate in the same manner that nonrelativistic, Galilei group Casimir invariants are only an approximate limiting case. Mass and spin are no longer the Casimir invariant eigenvalues labeling the irreducible representations. Actions of the representations of the general quaplectic can transform states with a given spin and mass into a state where these are different. The effects of the quaplectic group may become physically significant for strongly interacting systems where the relative forces between particle states approach b. These effects become apparent at the Planck scales λ_{α} defined in terms of the c, b and $\hbar$ basic dimensional constants. (All three of these constants are required in the quaplectic group unlike the Poincaré group for which only c appears.) Depending on the value of b, these effects may be difficult to access directly.

A possible calculation that would provide a test of the quaplectic symmetry may be obtained by noting that both free and interacting particle states are states in the Hilbert space of the representations of the quaplectic group. The Hilbert space of the representations of the Poincaré group are only states for free stable particle particles and not for interacting or decaying particles [23]. The usual Poincaré group is one of the four Poincaré groups that are subgroups of the quaplectic group. Therefore, we can consider a reduction of $\mathcal{Q}(1,n)$ with respect to $\mathcal{E}(1,n)$ (or its cover). Each irreducible representation of the quaplectic group is expected to contain a number of unitary irreducible representations of the Poincaré group labeled by mass and spin (μ, s). As the quaplectic group has discrete representations, one would expect these Poincaré irreducible representations also be discrete, (μ_k, s_k), $k = 1, 2, \ldots$. This would be a mass-spin spectrum and determine a discrete set of mass values μ_k for the free particle states in each unitary irreducible representation of the quaplectic group. We have definitive experimental information

about such discrete spectrums in the low energy regime against which to test this data. The generators that are not in the Poincaré subalgebra enable transitions between irreducible Poincaré particle states. That is, a free particle with one (discrete) mass and spin is transformed into another.

The mathematical problem of determining this embedding and its labelling is difficult. It has been solved for $\mathcal{SO}(3) \subset \mathcal{U}(3)$[24]. One approach may be to extend these to $\mathcal{SO}(1,n+1) \subset \mathcal{U}(1,n+1)$. It is well known that the $\mathcal{E}(1,n)$ is a group contraction limit of $\mathcal{SO}(1,n+1)$ and it is likewise true that $\mathcal{Q}(1,n)$ is a group contraction limit of $\mathcal{U}(1,n)$. Thus, it may be possible to obtain the desired embedding using these group contractions if the general embedding $\mathcal{SO}(1,n+1) \subset \mathcal{U}(1,n+1)$ can be solved.

The theory that has been discussed is a *special* or global theory. That is, it is the counterpart of the special relativity theory as apposed to the general relativistic theory. It should be possible to create a more general theory by lifting the identification of $\mathbb{Q}^{1,n}$ to the tangent space of a general, nonabelian manifold with curvature and making the parameters local. Interestingly, Schuller [13] proved a *no go* theorem for Hermitian metrics but the extra IU term required for it to be a Casimir invariant of the quaplectic group causes this *no go* theorem to not be applicable.

The next steps in this research are to show how the quaplectic group reduces to effectively the Poincaré case in the low interaction limit. This mathematically is the group contraction in the limit $b \to \infty$ just as the Poincaré group contracts to the Galilei group that is a semi-direct product with an Euclidian homogeneous group. This is being addressed in a follow-on paper [22]. A closely associated problem is to determine how the field equations (53), representations of the Casimir operators, reduce in this limit to effectively the usual Klein-Gordon, Dirac, Maxwell and so on of the Poincaré theory. Note the discrete *spectrum* on the right hand side of these equations. A first step in this difficult problem is to study the field equations of the compact case (55) to understand the spectrum of the *spinning* oscillator. The full field equations with nontrivial representations of the $\mathcal{SU}(1,n)$ involving relativistic spinning oscillators [25],[26],[27] need to be explicitly studied. Finally, the reduction or the quaplectic group with respect to the Poincaré group would give insight into the mass-spin spectrum as described above.

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